Theoretical and Numerical Study of Models of Entanglement for Neutrons

Candidate Number: 8221T
Supervisor: Dr Crispin Barnes

14 May, 2012

Abstract

We propose and investigate a scheme for detecting gravitational waves using the entanglement generated by the dynamics of a pair of neutrons trapped in a harmonic well. We develop a model that combines the effects due to plane gravitational wave solutions of the linearised field equations in the weak-field limit of general relativity with the time-dependent Schrödinger equation. Numerical simulations show that entanglement amplifies the effect of high frequency gravitational waves on the quantum state. In the proposed experiment, for realistic wave amplitudes and frequencies, the final state is practically indistinguishable from one unaffected by gravitational radiation. However, the results also show that quantum entanglement could be used in the future for high frequency gravitational wave detection.

1 Introduction

A significant amount of experimental effort has gone into detecting gravitational waves since the 1960s. They were first predicted by Einstein in 1916 as a consequence of general relativity. Mass (or energy) warps spacetime and changes in the shape or position of such objects will cause distortions which propagate as waves at the speed of light. Gravitational waves have still not been observed directly. However, the study of the period of the binary pulsar discovered by Hulse and Taylor in 1974 provides strong indirect evidence for their existence [1]. The search for direct evidence of gravitational waves has resulted in a number of large-scale experiments such as the Laser Interferometer Gravitational-Wave Observatory (LIGO) and the Laser Interferometer Space Antenna (LISA). The main difficulty of direct detection of gravitational radiation is its small effect on a detector, distortions from equilibrium on Earth due to astrophysical sources are predicted to be no larger than one part in 10^{21} [2]. To observe such a small effect an extremely sensitive apparatus is necessary. The LIGO and LISA experiments use laser interferometry as a means to detect such tiny changes. However, the fragile nature of entanglement could provide an alternative for an experiment to detect this effect.

Entanglement, a phenomenon unique to quantum mechanics, allows for stronger correlations between separate components of a composite system than are possible with classical statistics. This “spooky action at a distance” led Einstein to dismiss the theory as an incomplete description of reality [3]. However, in 1964 John Bell showed that no physical theory of local hidden variables, as suggested by Einstein, can reproduce the predictions of quantum mechanics. A number of experiments have been performed to test Bell's theorem and all of them provide strong evidence for the validity of quantum mechanics. The first definitive experiment was performed by Alain Aspect in 1982 [4].

Experiments have shown that quantum entanglement is not only real, but that it can also be used as a resource. The idea that it can be generated and manipulated like any other physical property of a system gave rise to the field of quantum information. Entanglement has allowed us to exceed limits imposed by classical mechanics in computing, cryptography and data transmission [5]. However, in reality quantum entanglement is a fragile resource which is very difficult to control. Decoherence, the loss of quantum coherences due to coupling to the environment, occurs on time scales much shorter than the rate at which we can manipulate the systems experimentally [6]. This sensitivity to environmental effects is one of the main obstacles in developing quantum technologies and is a subject of active research. However, this fragility could potentially be used to measure very small effects that require extremely sensitive detectors.

We propose and investigate numerically the possibility of performing an experiment to detect gravitational waves using the entanglement between a pair of neutrons initially localized on either side of a harmonic potential in a multilayer. Entanglement is generated in collisions due to the particles' natural motion [7]. By working in the weak-field limit of general relativity we combine the effect of gravitational waves with the Schrödinger equation. The resulting equation is then investigated numerically and we demonstrate that entanglement amplifies the effect of a gravitational wave, but the effect is too small to detect using conventional, easily accessible techniques originally envisaged for this experiment. However, the results show that entanglement can be a useful mechanism for detecting high frequency waves.

In section 2, we present the experimental setup that will be investigated and the mechanism for entanglement generation. In section 3, we describe the effect of gravitational waves, the neutron-neutron interaction and how these elements are combined in a two-particle Hamiltonian. We also address various concerns that arise when combining general relativistic effects with quantum mechanics. Section 4 presents the numerical simulations of the modified Schrödinger equation and their implications for the feasability of the suggested experiment. We conclude in section 5.

2 The Experiment

2.1 Gravitational Waves

The strength of gravitational radiation on Earth from typical astrophysical sources is very small. The largest strains one might expect to measure are of order 10^{-21} [2]. Therefore, we make the physical assumption that the gravitational fields are weak. Mathematically this corresponds to linearising the gravitational field equations. The basic mathematical framework of gravitational waves in linearised general relativity is summarised in appendix A. It is possible to choose a gauge in which the coordinate separation between the particles is constant at all times, but this has no coordinate-invariant physical meaning [2].

However, the physical spatial separation l, which is given by $l^2 = -g_{ij}\xi^i\xi^j$, will vary since the metric tensor components are not constant in the presence of a gravitational wave. The effect of a single passing gravitational wave on a cloud of non-interacting test particles is illustrated in figure 1.

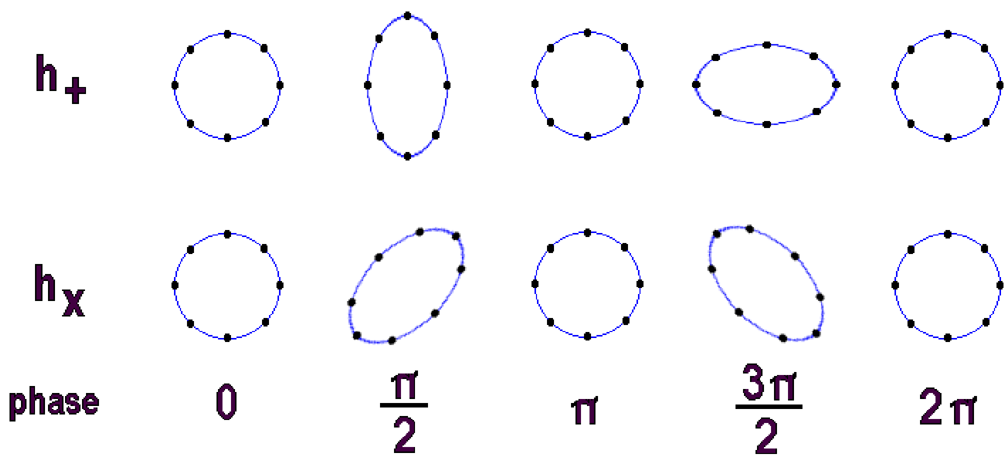

Figure 1: The effect of a passing gravitational wave on a circle of test particles. The two different polarisations are at 45° to each other. Figure reproduced with permission [8].

2.2 The Experimental Setup

We consider a one-dimensional system where the two neutrons with opposite spins interact in a quantum harmonic oscillator. The two particles are introduced in coherent states at rest on either side of the potential. They will oscillate in the harmonic trap, interact and eventually tunnel out. Repeating this experiment several times for the same initial condition will reveal information about the state and entanglement generated within the multilayer. If all unwanted effects are reduced to acceptable levels then the results will depend on the presence of gravitational waves if they have an effect on the entanglement of the two neutrons.

Such a trap can be constructed using a multilayer of ultrathin magnetic films of suitably chosen materials using molecular beam epitaxy. The detailed behaviour of neutrons in such multilayers is presented in [9] where polarized neutron reflection was used to study the magnetic properties of thin films. The potential energy in the αth region can be written as a sum of a nuclear and magnetic term

$$V_\alpha = \frac{\hbar^2}{2\pi m_n}\rho_\alpha b_\alpha - \boldsymbol{\mu_n}\cdot\boldsymbol{B_\alpha}, \tag{1}$$

where $\boldsymbol{\mu_\alpha}$, b_α, $\boldsymbol{B_\alpha}$ and ρ_α are the neutron moment, coherent nuclear scattering length, magnetic field due to the magnetization in the region and atomic density, respectively.

The harmonic oscillator potential itself can be formed from layers with no magnetization as long as the thickness of a single film is much less than the particle wavelength so the discrete nature of the material can be ignored. The second term in (1) leads to different values for the potential depending on the orientation of the neutron's spin relative to the magnetic field. Hence, we can use magnetized layers as spin dependent barriers. By placing such barriers at the two ends of the trap we provide means for a spin measurement.

It has been shown that under certain approximations the quantum dynamics of such a system are sufficient to generate maximally entangled states under appropriate conditions [7]. Producing maximally entangled states with high fidelity is not necessary in this experiment, but the work by Owen et al. provides a useful framework to describe and model the entanglement in a harmonic trap.

The two particles are initialized separately such that their single particle wave functions are spatially distinct at $t = 0$. We consider neutrons, mass m, in a harmonic potential of the form $V = \frac{1}{2}m\omega^2 x^2$ initially in the coherent states $\psi_{L/R}(x)$, where

$$\psi_R = A \exp\left(-\frac{m\omega}{2\hbar}(x - x_0)^2\right), \tag{2}$$

$$\psi_L(x) = \psi_R(-x), \tag{3}$$

$x_0 > 0$ and A is a normalization constant. To be spatially distinct, the single particle wave functions must satisfy $\int_{-\infty}^{0} \psi_R(x)\,\mathrm{d}x \to 0$ and $\int_0^\infty \psi_L(x)\,\mathrm{d}x \to 0$. We can then write the initial two-particle state as

$$\begin{aligned} |\Psi(t=0)\rangle &= \left(\int \psi_L(x_1) a^\dagger_{\sigma_1}(x_1)\,\mathrm{d}x_1\right)\left(\int \psi_R(x_2) a^\dagger_{\sigma_2}(x_2)\,\mathrm{d}x_2\right)|0\rangle \\ &\equiv \left(\int\int \Psi(x_1, x_2; t=0) a^\dagger_{\sigma_1}(x_1) a^\dagger_{\sigma_2}(x_2) \mathrm{d}x_1 \mathrm{d}x_2\right)|0\rangle \\ &\equiv |L_{\sigma_1} R_{\sigma_2}\rangle \end{aligned} \tag{4}$$

where $a^\dagger_{\sigma_i}(x_i)$ are the creation operators for a neutron at x_i with spin σ_i. We only consider neutrons which are introduced into the trap with opposite spins, $\sigma_1 \neq \sigma_2$, hence we can treat them as distinguishable. Therefore, the creation operators commute.

The two-particle Hamiltonian for the two neutrons in the same harmonic potential is given by

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) + \frac{1}{2}m\omega^2(x_1^2 + x_2^2) + V_{nn}(x_1 - x_2) \tag{5}$$

where $V_{nn}(x_1 - x_2)$ is a translationally invariant potential between the two neutrons and its exact form will be discussed in the next section.

In the parity-dependent harmonic approximation, which assumes that the spectrum of the Hamiltonian in (5) is equal to the spectrum of a quantum harmonic oscillator except for a parity-dependent energy shift, the wave function at integer half-periods has been shown to be

$$|\Psi(t = n\pi/\omega)\rangle \propto \cos\theta |L_{\sigma_1} R_{\sigma_2}\rangle + e^{i\chi}\sin\theta |R_{\sigma_1} L_{\sigma_2}\rangle, \tag{6}$$

where $\theta = n\pi\phi/2$, $\phi\hbar\omega$ is the energy shift to the even parity eigenstates and χ is some phase [7].

3 The Model

3.1 Gravitational Waves

Any reasonable signal possible to detect on Earth will be of astrophysical origin. Hence, we can assume that the source is far enough to treat the waves as plane waves. Therefore, in order for our model to be able to account for several gravitational waves we have to expand the formalism for the plane wave solutions presented in appendix A to a superposition of multiple waves. We adopt the viewpoint that $h_{\mu\nu}$ is a symmetric tensor field (under global Lorentz transformations) defined in quasi-Cartesian coordinates on a flat Minkowski background spacetime. Since we work with linearised general relativity we can easily obtain the solution by superposing the single plane wave solutions

$$\bar{h}^{\mu\nu} = \sum_j (A_j)^{\mu\nu} \exp(i(k_j)_\rho x^\rho), \tag{7}$$

where $A_j^{\mu\nu}$ are constant components of symmetric tensors and $(k_j)_\mu$ are the constant, real components of vectors. We assume the summation convention, but explicitly state the summation over the index j as it does not run over the coordinate indices, but over all plane waves present.

We consider a guage transformation of the same form as the transverse traceless gauge used in appendix A which is defined as

$$\bar{h}^{0a}_{TT} \equiv 0 \quad \text{and} \quad \bar{h}_{TT} \equiv 0, \tag{8}$$

where latin alphabet indices run over the spatial dimensions only. Furthermore the Lorenz gauge condition gives the constraints

$$\partial_0 \bar{h}^{00}_{TT} = 0 \quad \text{and} \quad \partial_a \bar{h}^{ab}_{TT} = 0. \tag{9}$$

Whilst we cannot consider this gauge to be transverse anymore since we are considering waves travelling in different directions, we will keep the labels since we are using exactly the same definition. This generalisation is straightforward, because of the linear nature of the field equations in a weak gravitational field. The field tensor transforms as

$$\bar{h}^{\mu\nu}_{TT} = \bar{h}^{\mu\nu} - \partial^\mu \xi^\nu - \partial^\nu \xi^\mu + \eta^{\mu\nu} \partial_\rho \xi^\rho, \tag{10}$$

where ξ^μ are four functions that define the gauge transformation and which must satisfy $\Box^2 \xi^\mu$ to preserve the Lorenz gauge. The solution in (7) is a linear superposition of waves with different wavevectors and Each of the components can be transformed into the TT gauge using some set of functions ξ_j^μ (see appendix A for details). Therefore, we can transform (7) using $\xi'^\mu = \sum_j \xi_j^\mu$ since the transformation (10) is linear in ξ^μ. Applying this transformation gives us the same constraints on the constant $A_j^{\mu\nu}$ components separately for all values of j. Therefore in the new gauge the coefficients must satisfy

$$(A_j)^{0a}_{TT} = 0 \quad \text{and} \quad ((A_j)_{TT})^\mu_\mu = 0 \tag{11}$$

and the Lorenz gauge conditions require that

$$(A_j)^{00}_{TT} = 0 \quad \text{and} \quad (A_j)^{ab}_{TT} k_b = 0. \tag{12}$$

Using these conditions we can construct a traceless tensor which satisfies the definition in (8). It is a superposition of waves of the form (7)

$$\bar{h}^{\mu\nu}_{TT} = \sum_j (A_j)^{\mu\nu}_{TT} \exp(i(k_j)_\rho x^\rho), \tag{13}$$

where $(A_j)^{0\mu}_{TT} = (A_j)^{\mu 0}_{TT} = 0$, the spatial components are given by

$$(A_j)^{ab}_{TT} = ((P_j)^a_c (P_j)^b_d - \frac{1}{2}(P_j)^{ab}(P_j)_{cd})(A_j)^{cd}, \tag{14}$$

$(P_j)_{ab} \equiv \delta_{ab} - (n_j)_a (n_j)_b$ and $(n_j)_a = (\hat{k}_j)_a$.

3.2 Neutron-neutron Interaction

Nucleon-nucleon scattering is fundamentally a many body problem governed by quantum chromodynamics. However, the strong interaction has a very short range ($\sim$ a few fm) and $kR \ll 1$, where k is the neutron wave vector and R the range of the potential. This means that we only have to consider s-wave scattering and we can approximate the scattering potential with a delta function $V_{nn} = U\delta(\vec{r}_1 - \vec{r}_2)$. To obtain the value of U we use the fact that for s-wave scattering the cross-section is given by $\sigma = 4\pi a_s^2$, where a_s is the scattering length which can be measured experimentally. Therefore, in the Born approximation we obtain

$$U = \frac{2\pi a_s \hbar^2}{m_n}, \tag{15}$$

where m_n is the mass of the neutron. Gravitational waves cause the distance between the two neutrons, $|\vec{r}_1 - \vec{r}_2|$, to oscillate. However, because the inter-particle interaction is in the form of a contact potential it remains unaffected in the presence of a passing wave. The physical spatial separation will only be zero if the coordinate separation vector will also be zero.

3.3 The Schrödinger Equation

The gravitational wave background is predicted to have no significant components at wavelengths shorter than a few km which is much larger than the lengthscales we are considering for the experiment. Therefore, combined with the fact that we are only considering weak gravitational fields, we can assume that in order to model the effect of gravitational waves on the quantum state of two neutrons in a harmonic trap we do not need to resort to a full quantum description of the interaction.

In order to incorporate the effect of passing waves on the quantum state we begin by considering the two-particle Schrödinger equation for the Hamiltonian given in (5)

$$i\hbar \frac{\partial \Psi(x_1, x_2; t)}{\partial t} = \left[-\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) + \frac{1}{2} m\omega^2 (x_1^2 + x_2^2) + V_{nn}(x_1 - x_2) \right] \Psi(x_1, x_2; t). \tag{16}$$

We have already shown that the form of V_{nn} is unaffected. The potential due to the interaction with the harmonic trap also remains unchanged as it does not depend on the physical separation between two points. A huge benefit of the gauge we have chosen to work in is the fact that only the spatial components of the metric are affected by the gravitational wave which means that the time derivative on the left hand side does not have to be modified. Only the spatial derivatives have to be considered in this situation.

For a general metric the Laplacian of a scalar field is given by

$$\nabla^2\phi = \frac{1}{\sqrt{|g|}}\partial_a\left(\sqrt{|g|}g^{ab}\partial_b\phi\right), \tag{17}$$

where g is the determinant of the metric tensor which in the TT gauge, to first order in h, is equal to unity. We now want to reduce the problem to only one spatial dimension to simplify the model. As this is only an investigation into the feasibility of such an experiment such a simplification is desirable as it reduces the necessary computational time required to obtain qualitatively the same results. The gravitational waves are very weak so we assume that confinement in the y and z directions is strong enough that the wavefunction remains in its ground state along those axes at all times. This means that we can ignore all second derivative terms apart from $\frac{\partial^2\Psi}{\partial x^2}$ since this is the only significant kinetic energy term. Furthermore, we will have terms of the form $\partial_a g^{ab}\partial_b\phi$. These terms can also be ignored, because $\partial_a g^{ab} \propto k_a$ and for any realistic setup the wavelength of the waves will be much larger than the size of the expriment making these terms insignificant. Therefore the only relevant terms that we are left with are

$$\nabla^2\Psi = g^{xx}\frac{\partial^2\Psi}{\partial x^2}, \tag{18}$$

where the effective form of g^{xx} is obtained from (13)

$$g^{xx} = 1 + \sum_i A_i^{xx}\cos(\Omega_i t + \phi_i), \tag{19}$$

Ω_i is the frequency of the ith wave, ϕ_i is the ith wave's phase at $x = 0$ and we have ignored phase variation along the trap axis, $k_x x$, since we consider wavelengths much larger than the dimensions of the well.

The most general form of the one-dimensional Laplacian in the TT gauge that preserves probability when used in the Schödinger equation is

$$\nabla^2\Psi = g^{xx}\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial g^{xx}}{\partial x}\frac{\partial\Psi}{\partial x}. \tag{20}$$

This imposes some additional restrictions on the components of the metric tensor. We again ignore second derivative terms due to confinement along the other axes. However, we must now require $g^{xy} = g^{xz} = 0$ for equation (20) to be correct. In order to investigate gravitational waves with shorter wavelengths we want to be able to use this equation for the Laplacian. However, under the TT and Lorenz gauge conditions the additional constraints also require $g^{xx} = 0$ unless the wavevectors are perpendicular to the x-axis. This in turn implies $\partial_x g^{xx} = 0$ since $k_x = 0$. Therefore, in our investigation we always use equation (18), but for wavelengths comparable to or smaller than the trap dimensions this limits us to the study of waves perpendicular to the x-axis.

Neutrons are spin-$\frac{1}{2}$ particles and so also we have to address the question of how the intrinsic angular momentum of the particles couples to the gravitational waves. The theoretical possibility of such coupling was first considered by Kobzarev and Okun in 1963 [10]. If we consider a Newtonian gravitational potential ϕ then the coupling to spin would take the form $H_{int} = A\vec{\sigma} \cdot \nabla\phi$, where A is an amplitude and $\vec{\sigma}$ is the particle spin. However, this violates the equivalence principle which states that the trajectory of a point mass in a gravitational field depends only on its initial position and velocity, and is independent of its composition. Therefore, we do not have to take into account any spin-gravity coupling since we are working in the weak field limit.

The final form of the two-particle Hamiltonian that we will be investigating is

$$\hat{H} = -g^{xx}\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) + \frac{1}{2}m\omega^2(x_1^2 + x_2^2) + V_{nn}(x_1 - x_2). \tag{21}$$

4 Numerical Simulations

We want to study the effect of gravitational waves on the entanglement and distinguishability of the final state from one unaffected by the radiation. The von Neumann entropy of entanglement,

$$S = -\mathrm{Tr}(\rho_1 \log_2 \rho_1) = -\mathrm{Tr}(\rho_2 \log_2 \rho_2), \tag{22}$$

where ρ_i is the reduced density matrix of particle i, is a standard measure of entanglement of two particles. The fidelity of quantum states is a suitable measure of their distinguishability. It is given by

$$\mathcal{F} = |\langle\psi_0|\psi_{gw}\rangle|, \tag{23}$$

where $|\psi_{gw}\rangle$ is the state calculated in the presence of a gravitational wave background and $|\psi_0\rangle$ is the state calculated in their absence. $\mathcal{F}^2$ is then just the probability of observing the outgoing particle in the state predicted for a flat spacetime and will always be equal to unity if $|\psi_{gw}\rangle = |\psi_0\rangle$.

We can distinguish two regimes for gravitational waves interacting with a two particle quantum system in a harmonic trap based on their relation to the interaction time, τ, which is the length of time during which the wave packets overlap is significant. Firstly, there are high frequency waves for which $\Omega \gg \tau^{-1}$. Figure 2a shows the evolution of fidelity over a single collision for a single wave of frequency $\Omega = 100\omega$. The biggest change in fidelity occurs during the collision itself when the wave packets overlap, interact and entangle. Outside the collision, when the overlap is small the fidelity simply oscillates with an amplitude that increases with the particle's speed. We clearly see that the effect of the gravitational wave is amplified in the interaction which leads to a permanent change in the value of $\mathcal{F}$. The change in fidelity is dominated by the change in the entanglement of the two particles due to the radiation. The low frequency regime is defined as $\Omega \ll \tau^{-1}$. Figure 2b clearly shows that such waves have a larger effect on the fidelity, but it is independent of the interaction. This effect is classical and the low fidelity is due to a change of the classical trajectory and final position of the particle in the harmonic potential. We will not consider low frequency waves in this investigation anymore since if we were intending on studying such waves we would not need to resort to a quantum mechanical model to describe their effect on the particles.

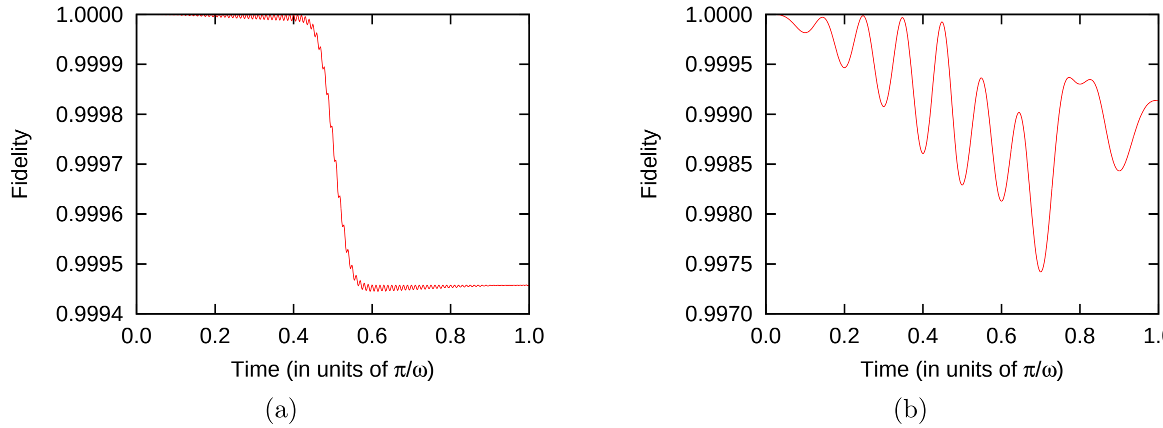

Figure 2: Fidelities of the wavefunction in the presence of a single gravitational wave at different frequencies to the unaffected state. (a) $\Omega = 100\omega$. The large drop at $t = \pi/\omega$ is evidence that interaction enhances the effect of the gravitational waves. (b) $\Omega = 10\omega$. We see no evidence of interaction at $t = \pi/\omega$ as it has no effect in this regime.

We see from figure 3 that the high frequency regime begins beyond 10ω as the inter-particle interaction starts having a visible effect on the fidelity of the final state. These results show that entanglement may be more useful in detecting high frequency gravitational waves rather than a general stochastic background since the effect of lower frequencies, which affect the classical behaviour of the system, are much larger. Figure 3 shows that in the high frequency regime the quantum effect of the waves on the fidelity can be even 10^6 larger than the effect due to the change in the classical trajectory. This amplification is likely to be even stronger for higher frequencies when we would no longer be able to ignore the first derivative terms in equation (17). Unfortunately, the gravitational wave background is predicted not to have any significant components above 100 kHz, which will usually be smaller than or comparable to the value of ω for a multilayer trap. Therefore, entanglement is not a very useful mechanism for detecting background radiation.

Figure 4 shows the von Neumann entropy of entanglement for a single collision. The form of the entropy does not change at all as a function of gravitational radiation. However, its final value does vary in the presence of waves, but the effect is very small compared to the change due to the particle dynamics in the collision itself. In the high frequency regime the waves' effect on the entanglement is larger than their effect on the trajectories of the particles, therefore, the fidelity of the final state to the unaffected wavefunction is a better measure of the change in entanglement of the system than entropy itself.

The values of the S and $\mathcal{F}$ are plotted over a suitable range of interaction strengths in figure 5 for a single high frequency wave. Whilst the largest change in fidelity does not coincide with the case where the final state is maximally entangled, the fidelity changes the most when entanglement is produced. This suggests that the amplification of the effect is due to the wave's influence on the system's entanglement. Following the fidelity and von Neumann entropy for multiple collisions confirms the correlation between fidelity and entropy. The dynamics of particles will cause the entanglement to increase and decrease during the collisions and the change in fidelity follows the same trend.

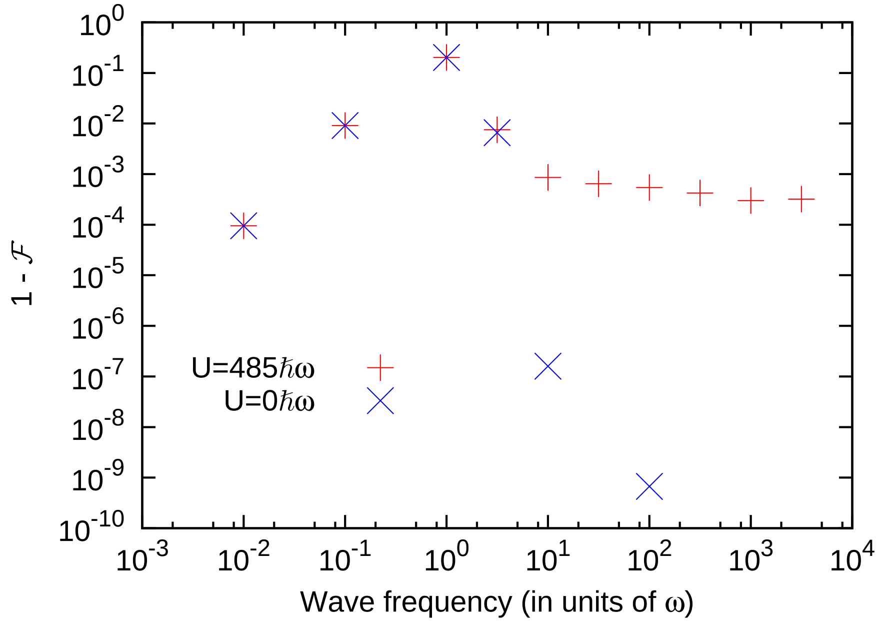

Figure 3: The fidelity change of the wavefunction in the presence of a single gravitational wave to an unaffected state for two different interaction strengths. We observe resonance at ω which is in the classical low frequency regime. The interplay between the gravitational waves and entanglement begins beyond $\Omega = 10\hbar\omega$.

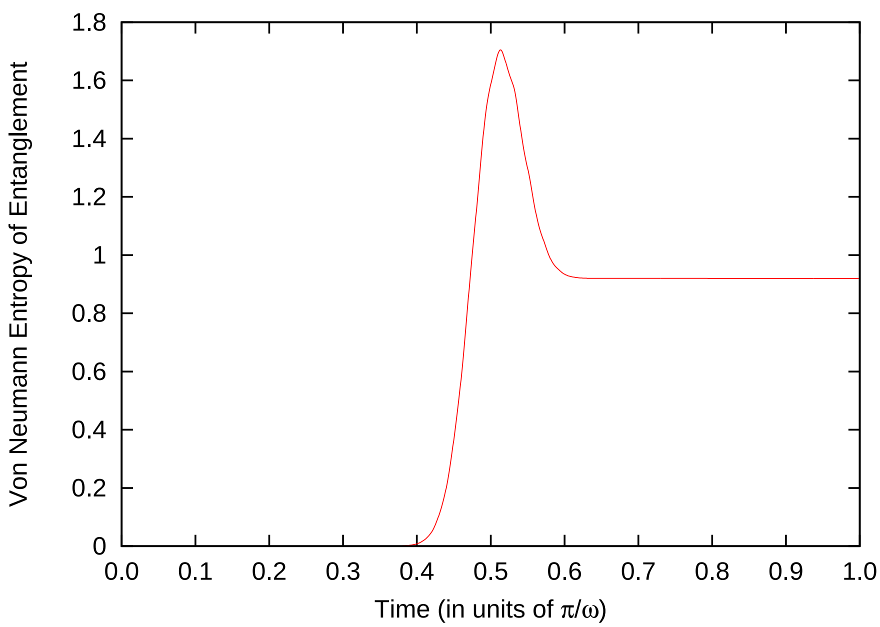

Figure 4: The von Neumann entropy of entanglement of the wavefunction in the presence of a single gravitational wave. Oscillations are present, but they are significantly smaller than the change in entropy due to the interaction.

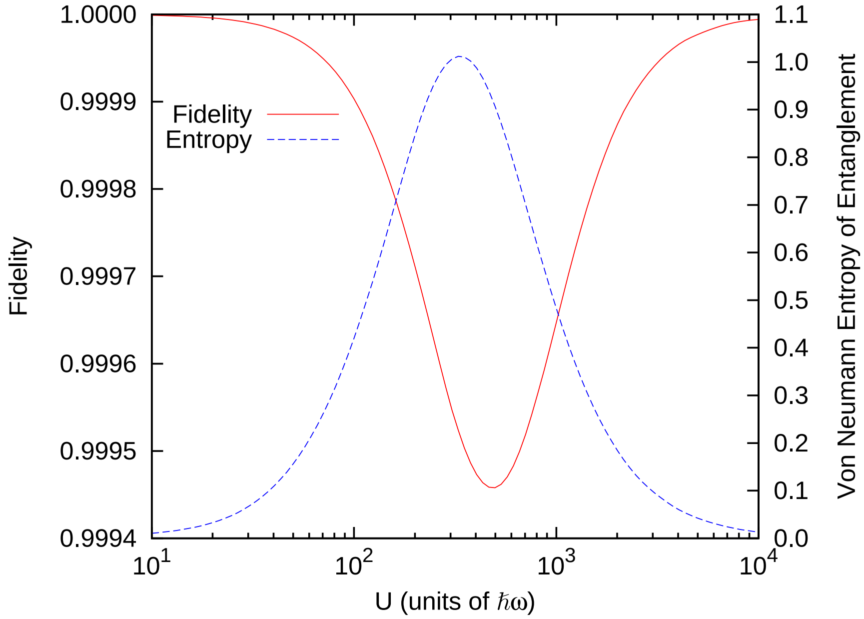

Figure 5: The values of the von Neumann entropy of entanglement and fidelity of the state affected by waves to the unaffected state after a single collision. Entanglement amplifies the effect of the gravitational wave on the wavefunction, but the maximum value of entropy does not coincide with the minimum value of fidelity.

To compare the effects of different radiation strengths on the quantum state we define the intensity of the signal to be $I = \sum_j (A_j^{xx})^2$ which is proportional to the energy flux of the wave [2]. All of the results produced so far were calculated for the largest possible value, $I = 0.01$. Higher intensities are not investigated since in those cases the weak field approximation is no longer valid. However, simulating realistic values of strain is beyond even machine double precision. In order to estimate the size of the effect of gravitational waves on the quantum state the value of fidelity is evaluated for different intensities that are within machine precision. The results in figure 6 show that the change in the final value of fidelity is proportional to the intensity of the waves. This is a very useful relationship as it lets us easily estimate the size of the gravitational waves' effect for different intensities. The square of the fidelity $\mathcal{F}^2$ is just the probability of observing the final state to be that predicted for a flat spacetime. Therefore, if we measure the final state (e.g. by measuring the spin of outgoing neutrons) several times then on average $1 - \mathcal{F}^2$ of the time we should obtain a different result than predicted by the dynamics of the particles alone in the absence of radiation. From the results in figure 6 we find that $1 - \mathcal{F} = 0.5I$. Therefore, the probability, p, of measuring a different state than expected can be shown to be

$$p = 1 - \mathcal{F}^2 \approx I. \tag{24}$$

Different wave combinations will lead to different numerical prefactors as the relationship between $1 - \mathcal{F}$ and I is linear regardless of the frequency regime and number of waves. The highest intensity we can investigate in the weak field limit and high frequency regime gives $p = 0.01$. This is a small value, but potentially measurable. However, extrapolating to the expected detectable values of strain gives $p \approx 10^{-40}$. Assuming Poissonian statistics in the spin counting process this requires $\sim 10^{27}$ measurements in order to make the error smaller than the signal which shows that such an experiment is impractical.

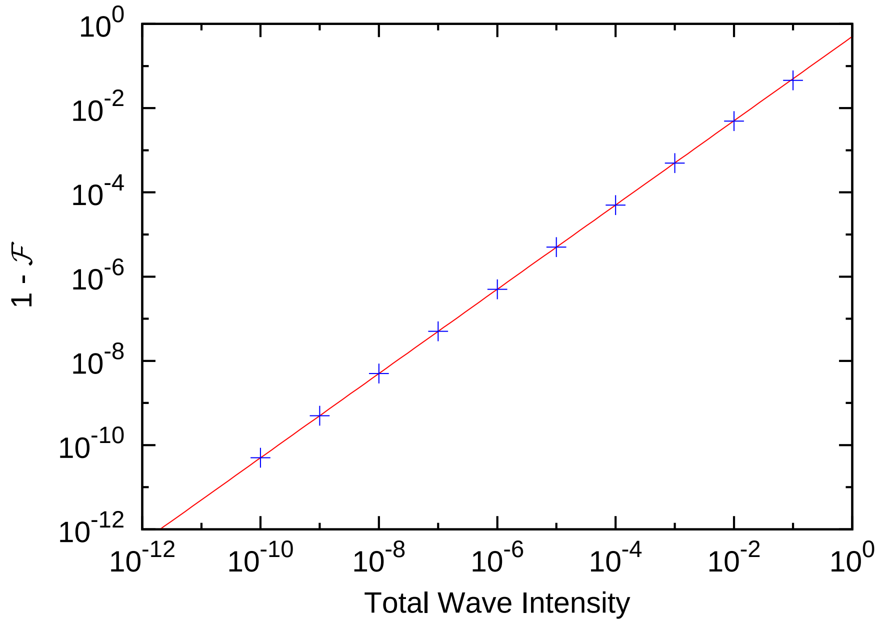

Figure 6: The change in fidelity $\Delta = 1 - \mathcal{F}$ plotted against the total intensity. This change is proportional to the total intensity $\Delta = 0.5I$ over a very large range of values. The relationship is linear regardless of the number of waves or the frequency components present.

We can now answer the question whether gravitational waves can be detected using neutrons in the suggested experiment. We have already discussed the applicability of the proposed scheme to detecting the gravitational wave background and concluded that the experiment's frequency range is too high and the classical effect of low frequency waves is much larger anyway. We have also shown that the probability of measuring an effect due to gravitational waves scales linearly with the wave intensity which means that for realistic wave amplitude values we will observe a different state only $\sim 10^{-40}$ of the time. Additionally, it can be shown that for typical multilayer dimensions and potentials the value of the neutron interaction in (15) will be $U \approx 10^7 \hbar\omega$ which corresponds to a point beyond the plotted range in figure 5. This means that two neutrons introduced in coherent states on either side of the well will not entangle in the collision and hence the quantum effect of even high frequency gravitational waves will be minimal. The neutrons will simply bounce of each other and behave classically. The most significant effect the waves have on such a system is to cause the physical separation of the two particles to oscillate and in this case two neutrons in a multilayer are not the most optimal system for detecting gravitational waves.

5 Conclusions

We have proposed and investigated the feasibility of an experiment to detect gravitational waves using the entanglement of two neutrons trapped in a harmonic well. The quantum dynamics of the two particles lead to entanglement which is affected by the presence of gravitational radiation. We have shown that entanglement amplifies the effect of high frequency gravitational waves on the wavefunction. However, for realistic values of the wave amplitudes the effect is too small to be measured in a device with the dimensions of a typical multilayer.

The effects of gravitational waves were combined with the quantum dynamics of the two neutrons in the weak-field limit, where the linearised field equation could be used. The effects of the oscillating metric were combined with the Schrödinger equation by modifying the kinetic energy term. The potential due to the harmonic well and the inter-nucleon interaction remained unchanged.

Numerical solutions to the modified time-dependent Schrödinger equation were obtained with the explicit staggered method. It was shown that there are two different behaviour regimes. For low frequency waves there are no additional quantum effects and the difference in the final quantum state is due to the different classical particle trajectories. These waves were not investigated as there are better ways of detecting them. However, the high frequency regime couples with the particle interaction and the effect is strongly dependent on its strength and is maximised close to the value for which the particles maximally entangle. This is an interesting result as it is a possible mechanism for detecting high frequency gravitational waves. However, for any experimentally accessible values we would not observe anything as the neutron-neutron interaction is too strong for any entanglement to be generated via the system's dynamics alone.

This experiment is not solely limited by the size of the signal. One other issue that would be difficult to resolve when building such an experiment is the isolation of the system from all environmental effects except for the gravitational waves. We have shown that the effect of radiation on the quantum state is extremely small and that entanglement is a key element, but with current technologies it is difficult to even maintain entangled resources for times longer than a few nanoseconds [11] unless we are working with trapped particles in ultra-high vacuum. Limiting undesirable decoherence is difficult with current technology and so any effects due to gravitational waves would be unobservable.

A Gravitational Waves in Linearised General Relativity

In general relativity we consider pseudo-Riemannian manifolds in which the interval is given by (assuming the summation convention)

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \tag{25}$$

where $g_{\mu\nu}$ are the components of the metric tensor field in the chosen coordinate system. A weak gravitational field corresponds to a region of spacetime that is only slightly curved. Thus, there exist coordinate systems x^μ in which the spacetime metric takes the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{26}$$

where $|h_{\mu\nu}| \ll 1$, the partial derivatives of $h_{\mu\nu}$ are also small and $\eta_{\mu\nu}$ are the components of the metric tensor in flat Minkowski spacetime.

It can be shown that for infinitesimal general coordinate transformations of the form

$$x'^\mu = x^\mu + \xi^\mu(x), \tag{27}$$

working to first order in small quantities, the metric transforms as follows:

$$g'_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu. \tag{28}$$

We note that $g'_{\mu\nu}$ is of the same form as (26), hence the new metric perturbation functions are related to the old ones via

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu. \tag{29}$$

We consider $h_{\mu\nu}$ to be a tensor field defined on the flat Minkowski background spacetime, hence (29) can be considered as analogous to a gauge transformation in electromagnetism. If $h_{\mu\nu}$ is a solution to the linearised gravitational field equations then the same physical situation is desribed by (29). However, in this viewpoint (29) is viewed as a gauge transformation rather than a coordinate transformation. We are still working in the same set of coordinates x^μ and have defined a new tensor whose components in this coordinate system are given by (29). Here we adopt the viewpoint that $h_{\mu\nu}$ is a symmetric tensor field (under global Lorentz transformations) defined in quasi-Cartesian coordinates on a flat Minkowski background spacetime.

The linearised field equations of general relativity can be written as

$$\Box^2 \bar{h}^{\mu\nu} = -2\kappa T^{\mu\nu}, \tag{30}$$

where $T^{\mu\nu}$ is the energy-momentum tensor, $\bar{h}^{\mu\nu} \equiv h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h$, $h = h^\mu_\mu$, provided that the $\bar{h}^{\mu\nu}$ components satisfy the Lorenz gauge condition

$$\partial_\mu \bar{h}^{\mu\nu} = 0. \tag{31}$$

In vacuo, where $T^{\mu\nu} = 0$, the general solution of the linearised field equations may be written as a superposition of plane-wave solutions of the form

$$\bar{h}^{\mu\nu} = A^{\mu\nu} \exp(ik_\rho x^\rho), \tag{32}$$

where $A^{\mu\nu}$ are constant components of a symmetric tensor and k_μ are the constant, real components of a vector. The Lorenz gauge condition is satisfied provided

$$A^{\mu\nu}k_\nu = 0. \tag{33}$$

Physical solutions may be obtained by taking the real part of (32).

Further gauge transformations will preserve the Lorenz gauge condition provided that the four functions $\xi^\mu(x)$ satisfy $\Box^2\xi^\mu = 0$. A common choice for plane gravitational waves is the transverse-traceless gauge defined by choosing

$$\bar{h}^{0i}_{TT} \equiv 0 \quad \text{and} \quad \bar{h}_{TT} \equiv 0, \tag{34}$$

where latin alphabet indices run over the spatial dimensions only. Furthermore the Lorenz gauge condition gives the constraints

$$\partial_0\bar{h}^{00}_{TT} = 0 \quad \text{and} \quad \partial_i\bar{h}^{ij}_{TT} = 0. \tag{35}$$

We note that the first condition in (35) implies that for non-stationary perturbations $\bar{h}^{00}_{TT}$ also vanishes. Therefore, in this gauge only the spatial components $\bar{h}^{ij}_{TT}$ are non-zero.

For a particular case of an arbitrary plane gravitational wave of the form (32) the conditions (34) imply that

$$A^{0i}_{TT} = 0 \quad \text{and} \quad (A_{TT})^\mu_\mu = 0 \tag{36}$$

and the Lorenz gauge conditions in (35) require that

$$A^{00}_{TT} = 0 \quad \text{and} \quad A^{ij}_{TT}k_j = 0. \tag{37}$$

Under these conditions the $A^{\mu\nu}_{TT}$ components are given by

$$A^{ij}_{TT} = (P^i_k P^j_l - \frac{1}{2}P^{ij}P_{kl})A^{kl}, \tag{38}$$

where $P_{ij} \equiv \delta_{ij} - n_i n_j$ and $n_i = \hat{k}_i$ [2].

B The Algorithm

We solve the two-particle Schrödinger equation for the Hamiltonian (21) via the explicit staggered method [12]. We choose our units such that $\hbar = 1$ and the neutron mass $m_n = 1$. The wave function is evaluated on a grid of discrete values for the independent variables

$$\Psi(x_1, x_2, t) = \Psi(x_1 = l\Delta x, x_2 = m\Delta x, t = n\Delta t) \equiv \Psi^n_{l,m}, \tag{39}$$

where l, m and n are integers. We separate it into real and imaginary parts,

$$\Psi^n_{l,m} = u^{n+1}_{l,m} + iv^{n+1}_{l,m}, \tag{40}$$

which allows us to solve the Schrödinger equation via a finite difference method with a pair of coupled equations:

$$u_{l,m}^{n+1} = u_{l,m}^{n-1} - \left\{ \alpha \left[v_{l+1,m}^{n} + v_{l-1,m}^{n} + v_{l,m+1}^{n} + v_{l,m-1}^{n} - 4v_{l,m}^{n} \right] - 2\Delta t V_{l,m} v_{l,m}^{n} \right\}, \tag{41}$$

$$v_{l,m}^{n+1} = v_{l,m}^{n-1} + \left\{ \alpha \left[u_{l+1,m}^{n} + u_{l-1,m}^{n} + u_{l,m+1}^{n} + u_{l,m-1}^{n} - 4u_{l,m}^{n} \right] - 2\Delta t V_{l,m} u_{l,m}^{n} \right\}, \tag{42}$$

where

$$\alpha = g^{xx} \frac{\Delta t}{\Delta x^2}, \tag{43}$$

and $V_{l,m}$ is the combined value of the potential due to the harmonic well and neutron-neutron interaction on the discretised grid given by

$$V_{l,m} = \frac{1}{2}\omega^2 (l^2 + m^2) \Delta x^2 + \nu \delta_{l,m}. \tag{44}$$

In order to evaluate the value of the real part at step $n+1$ we need to know the values of the real part at $n-1$ and the imaginary part at n. The same is true for evaluating the imaginary part. Therefore, we do not have to calculate both parts at every time step and we compute the real and imaginary parts at slightly different (staggered) times. The form of the discretised equations is identical to those presentied in [12]. However, the value of α is now scaled by the oscillating metric tensor component g^{xx}. This novel modification does not lead to instabilities and simulations have been confirmed to conserve probability to the same precision as before.

For the more general Laplacian (20), the equations have to be modified even further and new terms are introduced for the wavefunction's first derivative terms present

$$\begin{aligned} u_{l,m}^{n+1} &= u_{l,m}^{n-1} - \left\{ \alpha \left[v_{l+1,m}^{n} + v_{l-1,m}^{n} + v_{l,m+1}^{n} + v_{l,m-1}^{n} - 4v_{l,m}^{n} \right] \right. \\ &\quad \left. + \beta \left[v_{l+1,m}^{n} - v_{l-1,m}^{n} + v_{l,m+1}^{n} - v_{l,m-1}^{n} \right] - 2\Delta t V_{l,m} v_{l,m}^{n} \right\}, \end{aligned} \tag{45}$$

$$\begin{aligned} v_{l,m}^{n+1} &= v_{l,m}^{n-1} + \left\{ \alpha \left[u_{l+1,m}^{n} + u_{l-1,m}^{n} + u_{l,m+1}^{n} + u_{l,m-1}^{n} - 4u_{l,m}^{n} \right] \right. \\ &\quad \left. + \beta \left[u_{l+1,m}^{n} - u_{l-1,m}^{n} + u_{l,m+1}^{n} - u_{l,m-1}^{n} \right] - 2\Delta t V_{l,m} u_{l,m}^{n} \right\}, \end{aligned} \tag{46}$$

where

$$\beta = \frac{\partial g^{xx}}{\partial x} \frac{\Delta t}{2\Delta x}. \tag{47}$$

The two components can still be evaluated at staggered times. This further modification does not cause instabilities and conserves probability very well.

C Decoherence

Decoherence is the process through which quantum correlations disperse information throughout the degrees of freedom that are effectively beyond the observer's control. It has been suggested as the mechanism through which a measurement of a quantum system is effectively made [6]. In general, any interaction with the environment will eventually lead to decoherence and this is one of the main obstacles in generating and maintaining entangled resources. At the beginning of this investigation we proposed that entanglement may be sensitive enough to be affected by gravitational waves. In our approach we have only investigated the dynamics of two neutrons in a harmonic well subject to gravitational radiation by solving the Schrödinger equation for the modified Hamiltonian (21). This has allowed us to study the effects of the waves on the wavefunction of neutrons entangling in a harmonic potential, but the fact that we keep track of all of the degrees of freedom meant that it was impossible to observe decoherence.

In order to be able to describe decoherence we have to construct a dynamical model of a system coupled to its environment. In the Feynman-Vernon theory of the influence functional [13] we consider a system A interacting with a second system B (the reservoir) described by the Hamiltonian

$$H = H_A + H_I + H_B, \tag{48}$$

where system A consists of a single particle, the reservior consists of N particles and H_I is the interaction Hamiltonian. We can obtain the reduced density operator for system A by tracing out all the environment coordinates. Assuming that the initial density operator is separable into a product of the density operators for A and B we get

$$\rho(x,y,t) = \int \mathrm{d}x'\mathrm{d}y' J(x,y,t;x',y',0)\rho_A(x',y',0), \tag{49}$$

where x and y are the position coordinates of the single particle,

$$J(x,y,t;x',y',0) = \int\int \mathrm{D}x\mathrm{D}y \exp\left[i\frac{S_A[x]}{\hbar}\right]\exp\left[-i\frac{S_A[y]}{\hbar}\right]F[x,y], \tag{50}$$

$F[x,y]$ is the so called influence functional given by

$$\begin{aligned} F[x,y] \quad &= \quad \int \mathrm{d}\boldsymbol{R'}\mathrm{d}\boldsymbol{Q'}\mathrm{d}\boldsymbol{R}\rho_B(\boldsymbol{R'},\boldsymbol{Q'},0)\int\int \mathrm{D}\boldsymbol{R}\mathrm{D}\boldsymbol{Q} \\ &\quad \times \exp\left[\tfrac{i}{\hbar}\left(S_I[x,\boldsymbol{R}] - S_I[y,\boldsymbol{Q}] + S_B[\boldsymbol{R}] - S_B[\boldsymbol{Q}]\right)\right], \end{aligned} \tag{51}$$

S_i is the action corresponding to the Hamiltonian H_i and $\boldsymbol{R}$, $\boldsymbol{Q}$ are vectors of the positions of the reservoir particles. This has been solved exactly in [13] in the limit of a weak coupling, where we only have to consider the linear response of the reservoir to the system and we can describe the environment in the harmonic approximation.

The gravitational wave background could potentially be modelled with a collection of harmonic oscillators. However, as we have seen in section 3 the interaction of the wave function with gravitational waves is implemented through a modification of the kinetic energy term and not via an effective potential and modelling the coupling with the system with a linear response would be insufficient. The more complicated interaction would lead to an intractable form for the influence functional. Furthermore, there is no way of obtaining an irreversible process such as decoherence using the above solution. We do not consider a reservoir of infinite size and any information lost by the system will return within a finite period of time.

Caldeira and Leggett addressed the issue of irreversibility in 1983 [14] in an attempt to solve the problem of quantum Brownian motion. Instead of coupling the system A to a reservoir B they replaced the system-environment coupling with an externally applied classical force $F(t)$. The fluctuating force has to obey the standard stochastic relations

$$\langle F(t)\rangle = 0, \tag{52}$$

$$\langle F(t)F(t')\rangle = 2\eta k_B T\delta(t-t'), \tag{53}$$

where η is a damping constant, k_B is Boltzmann's constant and T is the temperature. However, there are issues with applying the Caldeira-Leggett model to our problem regarding the nature of the effect produced by the waves. In the situation presented in figure 1 all of the test particles have constant spatial coordinates, they are stationary. However, the measured separation between the points changes according to $l^2 = -g_{ij}\xi^i\xi^j$. Modelling this effect via a classical force is questionable.

In order to describe decoherence a master equation approach is necessary, but the most successful models for dissipation due to a system's interaction with the environment are not appropriate for the interaction of a quantum wave function with gravitational waves. The environment has also been modelled with a quantum field [15] and for a certain class of interactions the approach is equivalent to the Caldeira-Leggett model. If a quantum field theory of gravity was developed then a similar approach could be used to derive a master equation to model decoherence due to gravitational fields.

References

[1] J.M. Weisberg, J. H. Taylor, *Relativistic Binary Pulsar B1913+16: Thirty Years of Observations and Analysis*, arXiv:astro-ph/0407149v1 (2004).

[2] M. P. Hobson, G. Efstathiou, A. N. Lasenby, *General Relativity: An Introduction for Physicists*, (ch. 17-18), Cambridge University Press, 2009. ISBN-13 978-0-521-82951-9.

[3] A. Einstein, B. Podolsky, N. Rosen, *Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?* Phys. Rev. 47, 777780 (1935).

[4] A. Aspect, J. Dalibard, G. Roger, *Experimental Test of Bell's Inequalities Using Time-Varying Analyzers* Phys. Rev. Lett. 49, 1804-1807 (1982).

[5] A. Steane, *Quantum Computing*, arXiv:quant-ph/9708022v2 (1997).

[6] W. H. Zurek, *Decoherence and the transition from quantum to classical – REVISITED* arXiv:quant-ph/0306072v1 (2003).

[7] E. T. Owen, M. C. Dean, C. H. W. Barnes, *Generation of Entanglement between Qubits in a One-Dimensional Harmonic Oscillator*, Phys. Rev. A 85, 022319 (2012).

[8] http://www.johnstonsarchive.net/relativity/pictures.html

[9] S. J. Blundell, J. A. C. Bland, *Polarized Neutron Reflection as a Probe of Magnetic Films and Multilayers*, Phys. Rev. B 46, 3391-3400 (1992).

[10] I.Y. Kobzarev, L.B. Okun, *Gravitational Interaction of Fermions*, Soviet Journal of Experimental and Theoretical Physics, Vol. 16, p.1343 (1963).

[11] T. Fujisawa, T. Hayashi, H. D. Cheong, Y. H. Jeong, Y. Hirayama, *Rotation and Phase-shift Operations for a Charge Qubit in a Double Quantum Dot*, Phys. Rev. Lett 91 226804-1 (2003).

[12] J. J. V. Maestri, R. H. Landau, M. J. Paez, *Two-Particle Schroedinger Equation Animations of Wavepacket-Wavepacket Scattering (revised)*, arXiv:physics/9909042v1 [physics.comp-ph] (2008).

[13] R. P. Feynman, F. L. Vernon Jr., *The theory of a general quantum system interacting with a linear dissipative system*, Annals Phys. 24, 118-173 (1963).

[14] A. O. Caldeira, A. J. Leggett, *Path Integral Approach to Quantum Brownian Motion*, Physica 121A, 587-616 (1983).

[15] W. G. Unruh, W. H. Zurek, *Reduction of a wave packet in quantum Brownian motion*, Phys. Rev. D. 40, 1071-1094 (1984).